REAL ANALYSIS

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For any real number x, $|x| = x$ if $x \geq 0$ and $= -x$ if $x < 0$. It is clear that $|x| \geq 0$ for all x. Also $|x| = 0$ if and only if $x = 0$.

Properties: The absolute value functions has the following properties:

(i) $-|a| \leq a \leq |a|$ for every real number a.
(ii) $|ab| = |a||b|$ for all real numbers a, b.
(iii) $|-a| = |a|$ for all real numbers a.
(iv) Let $c > 0$. Then $|a| < c \iff -c < a < c$ and $|a| \leq c \iff -c \leq a \leq c$

Triangle inequality: For any two real numbers a and b, $|a + b| \leq |a| + |b|$.

Proof: Let a and b be two real numbers. Then by property (i), we have

$$-|a| \leq a \leq |a| \quad \text{and} \quad -|b| \leq b \leq |b|.$$

Now adding, we get

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Finally, applying property (iv), we get

$$|a + b| \leq |a| + |b|.$$

Reverse Triangle inequality: For any two real numbers a and b, $\Big||a| - |b|\Big| \leq |a - b|$.

Proof: Let a and b be two real numbers. Then by triangle inequality, we have

$$|a| = |a - b + b| \leq |a - b| + |b| \quad \text{and } |b| \leq |a - b| + |a|.$$

This two inequalities can be written as

$$|a| - |b| \leq |a - b| \quad \text{and } |b| - |a| \leq |a - b|$$

which yields that

$$-|a - b| \leq |a| - |b| \leq |a - b|.$$

Hence again by property (iv), we have

$$\Big||a| - |b|\Big| \leq |a - b|.$$

$$\Big||a| - |b|\Big| \leq |a - b| \leq |a| + |b|.$$

Problem: Let A and B be two nonempty sets of real numbers and let L be a real number. Suppose that there exists positive real numbers x, y such that $|a - L| < x$ for all $a \in A$ and that $|2b - L| < y$ for all $b \in B$. Show that $|a - 4b| < x + 2y + |L|$ for all $a \in A$ and $b \in B$.

Solution: Let $a \in A$ and $b \in B$. Then we have

$$|a - 4b| = |a - L + 2L - 4b - L| \leq |a - L| + 2|2b - L| + |L| < x + 2y + |L|.$$

Problem: Let S be a nonempty set of real numbers and let a be a nonzero real number. Suppose that $|x - a| < \frac{|a|}{2}$ for all $x \in S$. Prove that $|x| > \frac{|a|}{2}$ for all $x \in S$.

Problem: Let x be a real number. If $|x| < \varepsilon$ for each $\varepsilon > 0$, then $x = 0$.

Solution: We will prove this by method of contradiction. Suppose that $x \neq 0$. Let $\varepsilon = \frac{|x|}{2}$. Then $\varepsilon > 0$. Note that $|x| > \varepsilon$- a contradiction to the hypothesis that $|x| < \varepsilon$ for each $\varepsilon > 0$. It follows that $x = 0$.

Problem: Let x and a be real numbers. Suppose that $x < a + \varepsilon$ for all positive numbers ε. Prove that $x \leq a$.

Problem: Let a and b be two real numbers such that $|a - b| < \varepsilon$ for each $\varepsilon > 0$. Show that $a = b$.

Interval: A set S of real numbers is an interval if and only if S contains at least two points and for any two points $x, y \in S$, every real number between x and y also belongs to S i.e., $\{z : x \leq z \leq y\} \subset S$.

Example: Consider the set $S = \mathbb{Q}$ or $S = \mathbb{Z}$ or $S = \mathbb{N}$. None of them are intervals.

Example: Consider the set $S = \{x : a < x < b\} = (a, b)$. It is called open interval. $(-\infty, a]$. $(-\infty, \infty)$.

$(0, 1) \cup (2, 3)$.

Boundedness: Let S be a set of real numbers. Then S is said to be **bounded above** if there is a real number M such that $x \leq M$ for all $x \in S$. The number M is called an **upper bound** of the set S.

The sets $\mathbb{N}, \mathbb{Z}$ are not bounded above. The set $S = (-\infty, 2021)$ is bounded above, since there is M with $M \geq 2021$ we have $x \leq M$ for all $x \in S$.

Let S be a set of real numbers. Then S is said to be **bounded below** if there is a real number m such that $x \geq m$ for all $x \in S$. The number m is called an **lower bound** of the set S.

The set $\mathbb{N}$ is bounded below but not bounded above. The set $\mathbb{Z}$ is not bounded below.

A set S is said to be bounded if it is both bounded below and bounded above. In this case, there is a real number $M > 0$ such that $|x| \leq M$ for all $x \in S$ (Verify!) i.e., if $\exists M > 0$ such that $\forall x \in S$, we have $|x| \leq M$.

A set which is not bounded is said to be unbounded. In this case, $\forall M > 0$ $\exists x \in S$ such that $|x| > M$.

Supremum or least upper bound: A real number M is said to be the supremum of a non-empty set S if

(1) M is an upper bound of S,

(2) no number less than M is an upper bound of S i.e., for any $\varepsilon > 0$, there is an element $y \in S$ such that $y > M - \varepsilon$.

In this case we write, $\sup S = M$.

For example, let $S = (0, 1)$. Then note that 2 is an upper bound of S but it is not supremum of S. It is to be noted that 1 is the supremum of S.

Let us take another example. Let $S = \mathbb{N}$. Then $\sup \mathbb{N} = \infty$.

Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\sup S = 1$.

infemum or greatest lower bound:

Exercise: Prove that a finite set always contains its supremum and infemum.

Question: Does every set that is bounded above have a supremum?

Let us consider the set $S = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$. Clearly S is non-empty set of rational numbers that is bounded above. Note that any positive rational number y such that $y^2 > 2$ is an upper bound of S.

Claim: S does not have any supremum in the set of rational numbers. Let $p \in \mathbb{Q}$ be the supremum of S and let

$$q = p - \frac{p^2 - 2}{p + 2}.$$

Then q is also a rational number. Now,

$$q^2 - 2 = \frac{(2p+2)^2}{(p+2)^2} - 2 = \frac{2(p^2-2)}{(p+2)^2}.$$

Using the equations for q and $q^2 - 2$ reveals the following:

$$\text{if } p^2 < 2, \text{ then } q > p \text{ and } q^2 < 2$$

$$\text{if } p^2 > 2, \text{ then } q < p \text{ and } q^2 > 2$$

Completeness axiom: Each non-empty set of real numbers that is bounded above has a supremum.

Archimedian property of real numbers: If a and b are positive real numbers, then there exists a positive integer n such that $na > b$.

Application: for any $\varepsilon > 0$, there is a natural number N such that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

for all $n \geq N$.

Let $\varepsilon > 0$. Then by Archimedian property, there is a positive integer N such that

$$N\varepsilon > 1$$

which implies that

$$\frac{1}{N} < \varepsilon.$$

Hence there is positive integer N such that

$$\frac{1}{n} < \varepsilon$$

for all $n \geq N$. This proves that the sequence $\{\frac{1}{n}\}$ is convergent and converges to 0.

Proof of Archimedian property: The result is trivial if $a \geq b > 0$. Let us assume that $0 < a < b$. We proof the result by contradiction. Suppose there is no positive integer n such that $na > b$. Then $na \leq b$ for every positive integer n. This imply that the set $\{na : n \in \mathbb{N}\}$ is bounded above by b. By the completeness property, the set S has a supremum, say, M. Note that, $M - a < M$. Now by the definition of supremum, the number $M - a$ is not an upper bound of S. Cosequently, there is a positive integer k such that $ka > M - a$ which implies that

$$M < ka + a = (k+1)a.$$

Since $(k+1)a \in S$-this contradicts the fact that $M = \sup S$. Hence there is a positive integer n such that $na > b$.

By definition there are real numbers k, K such that $k \leq x \leq K$ for all $x \in S$. Let $M = \max\{|k|, |K|\}$. We have $x \geq k$ and so $-x \leq -k \leq |k| \leq M$ which implies that $x \geq -M$. Also $x \leq K \leq M$. Hence we have

$$-M \leq x \leq M \Longrightarrow |x| \leq M$$

for all $x \in S$ (Verify!).

Also note that for all $x \in S$,

$$-M \leq x \leq M \Longrightarrow x \in [-M, M]$$

i.e., $S \subset [-M, M]$.

Exercise: Let S be a set bounded above. Prove that

(a) for any $\lambda > 0$, $\sup \lambda S = \lambda \sup S$.

(b) for any $\lambda < 0$, $\inf \lambda S = \lambda \sup S$.

Solution: (a) Let $\lambda > 0$ and let $\sup S = M$. Then

$$x \leq M$$

for all $x \in S$ and therefore

$$\lambda x \leq \lambda M \tag{1}$$

This implies that λM is an upper bound of λS.

Now, let $\varepsilon > 0$ be arbitrary. Then there exists an element $y \in S$ such that

$$y > M - \frac{\varepsilon}{\lambda}$$

which implies that

$$\lambda y > \lambda M - \varepsilon \tag{2}$$

From (1) and (2) we find that λM is the least upper bound of λS i.e.,

$$\sup \lambda S = \lambda M = \lambda \sup S.$$

(b) Let $\lambda < 0$ and let $\sup S = M$. Then

$$x \leq M$$

for all $x \in S$ and therefore

$$\lambda x \geq \lambda M \tag{3}$$

This implies that λM is a lower bound of λS.

Now, let $\varepsilon > 0$ be arbitrary. Then there exists an element $y \in S$ such that

$$y > M - \frac{\varepsilon}{-\lambda}$$

which implies that

$$\lambda y < \lambda M + \varepsilon \tag{4}$$

From (3) and (4) we see that λM is the greatest lower bound of λS i.e.,

$$\inf \lambda S = \lambda M = \lambda \sup S.$$

Exercise: Let S be a set bounded below. Prove that

(a) for any $\lambda > 0$, $\inf \lambda S = \lambda \inf S$.
(b) for any $\lambda < 0$, $\sup \lambda S = \lambda \inf S$.

Solution: (a) Let $\lambda > 0$ and let $\inf S = m$. Then

$$x \geq m$$

for all $x \in S$ and therefore

$$\lambda x \geq \lambda m \tag{1}$$

This implies that λm is a lower bound of λS.

Now, let $\varepsilon > 0$ be arbitrary. Then there exists an element $y \in S$ such that

$$y < m + \frac{\varepsilon}{\lambda}$$

which implies that

$$\lambda y < \lambda m + \varepsilon \tag{2}$$

From (1) and (2) we see that λm is the greatest lower bound of λS i.e.,

$$\inf \lambda S = \lambda m = \lambda \inf S.$$

(b) Let $\lambda < 0$ and let $\inf S = m$. Then

$$x \geq m$$

for all $x \in S$ and therefore

$$\lambda x \leq \lambda m \tag{3}$$

This implies that λm is an upper bound of λS.

Now, let $\varepsilon > 0$ be arbitrary. Then there exists an element $y \in S$ such that

$$y < m + \frac{\varepsilon}{-\lambda}$$

which implies that

$$\lambda y > \lambda m - \varepsilon \tag{4}$$

From (3) and (4) we see that λm is the least upper bound of λS i.e.,

$$\sup \lambda S = \lambda m = \lambda \inf S.$$

Exercise: Let S be a bounded set. Prove that

(1) for any $\lambda > 0$, $\sup \lambda S = \lambda \sup S$, $\inf \lambda S = \lambda \inf S$.
(2) for any $\lambda < 0$, $\inf \lambda S = \lambda \sup S$, $\sup \lambda S = \lambda \inf S$.

Exercise: Let S be a bounded set and let A be a subset of S. Then prove that

$$\inf S \leq \inf A \leq \sup A \leq \sup S.$$

Solution: Let $\sup S = M$. Then

$$x \leq M$$

for all $x \in S$ and hence for all $x \in A$ as $A \subset S$. Therefore, M is an upper bound of A. Hence

$$\sup A \leq M = \sup S.$$

Now, let $\inf s = m$. Then

$$x \geq m$$

for all $x \in S$ and hence for all $x \in A$. This shows that m is a lower bound for A. Then

$$\inf A \geq m = \inf S.$$

Finally,

$$\inf S \leq \inf A \leq \sup A \leq \sup S.$$

Consequences of Archimedian property:

(1) Given $x > 0$, there exists $n \in \mathbb{N}$ such that $x > \frac{1}{n}$.

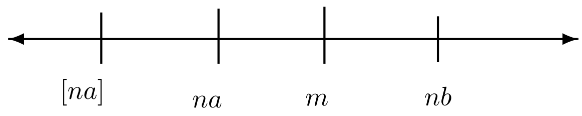
Solution: Apply Archimedian property taking $b = 1$ and $a = x$. Then there is $n \in \mathbb{N}$ such that $nx > b = 1$. Hence $x > \frac{1}{n}$.

(2) The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Solution: We prove this result by contradiction. Assume that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Then by completeness axiom, $\mathbb{N}$ has a supremum, say, M. Then for each $k \in \mathbb{N}$, we have $k \leq M$. Now, note that $M - 1 < M$ and therefore, $M - 1$ is not an upper bound of $\mathbb{N}$ and hence there exists $N \in \mathbb{N}$ such that $N > M-1$. This implies that $N+1 > M$. Since $N+1 \in \mathbb{N}$, we conclude that M is not an upper bound of $\mathbb{N}$-a contradiction. This contradiction proves that our assumption is wrong. Thus $\mathbb{N}$ is not bounded above.

(3) **Density property of $\mathbb{Q}$:** Given $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Solution: Since $b - a > 0$, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$. Let $k = [na]$ and $m = k + 1$. Then clearly, $na < m$. Now, we claim that $m < nb$.



Suppose that $m \geq nb$. Then

$$1 = (k+1) - k = m - k \geq nb - na = n(b-a) > 1$$

-a contradiction. Hence we have $m < nb$. Thus, we have $na < m < nb$ or $a < \frac{m}{n} < b$.

(4) **Density property of $\mathbb{I}$:** Given $a, b \in \mathbb{R}$ with $a < b$, there exists $t \in \mathbb{I}$ such that $a < t < b$.

Solution: Consider the real numbers $a - \sqrt{2} < b - \sqrt{2}$ and apply the result (3). Let us work out the details.

By the density of rational numbers, there is $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. This implies that $a < r + \sqrt{2} < b$. we claim that $r + \sqrt{2}$ is an irrational. Let us suppose that $s = r + \sqrt{2} \in \mathbb{Q}$. It follows that $s - r = \sqrt{2} \in \mathbb{Q}$-this is a contradiction. Hence the result follows.

(5) For each real number x, there exists an integer n such that $n \leq x < n+1$.

Solution: Let $S = \{k \in \mathbb{Z} : k \leq x\}$. We claim that $S \neq \phi$. If $S = \phi$, then we must have $p > x$ for all $p \in \mathbb{Z}$. Let $t \in \mathbb{N}$ be arbitrary. Set $p = -t \in \mathbb{Z}$ and therefore $-t = p > x$. It follows that $t < -x$. Since $t \in \mathbb{N}$ is arbitrary, it follows that $-x$ is an upper bound of $\mathbb{N}$ - a contradiction. Thus we must have $S \neq \phi$.

It is to be noted that x is an upper bound of S. Then by completeness property, S has a supremum, say, M. Then $M - 1$ is not an upper bound of S. So there exists $n \in S$ such that $n > M - 1$ or $n + 1 > M$. Also note that $n \leq x$. We claim that $n + 1 > x$. If not, then $n + 1 \leq x$ and so $n + 1 \in S$. Therefore $n + 1 \leq M$ - a contradiction. Hence we get $n \leq x < n + 1$.

Exercise: Let s and t be real numbers such that $s - t > 1$. Prove that there exists an integer n such that $t < n < s$.

Hint: Since $s - t > 1$, we have

$$s > t + 1 \geq [t+1] = [t] + 1 > t$$

as $[t] + 1$ is an integer. So it is enough to choose $n = [t] + 1$.

Exercise: Prove that $\sup(0,1) = 1$ and $\inf(0,1) = 0$.

Solution: It is to be noted that 1 is an upper bound of $(0,1)$. Let M be another upper bound of $(0,1)$. We claim that $M \geq 1$. If not, the we have $M < 1$. Then there exists $n \in \mathbb{N}$ such that $1 - M > \frac{1}{n}$ or $M < 1 - \frac{1}{n}$ and $1 - \frac{1}{n} \in (0,1)$ - this is contradiction to the fact that M is an upper bound of $(0,1)$. Then we must have $M \geq 1$. Hence $\sup(0,1) = 1$.

For the next part, note that 0 is a lower bound of $(0,1)$. Let m be a lower bound of $(0,1)$. We claim that $m \leq 0$. If not, then $m > 0$. Then there exists $n \in \mathbb{N}$ such that $m > \frac{1}{n}$ and $\frac{1}{n} \in S$ - a contradiction to the fact that m is a lower bound of $(0,1)$. Then we must have $m \leq 0$. Hence $\inf(0,1) = 0$.

Exercise: Let S be a bounded set. Prove that

$$\sup S - \inf S = \sup\{x - y : x, y \in S\}.$$

Solution: Let $\sup S = M$ and $\inf S = m$. Then $x \leq M$ for all $x \in S$ and $y \geq m$ or $-y \leq -m$ for all $y \in S$. Adding we get, $x - y \leq M - m$ for all $x, y \in S$. This shows that $M - m$ is an upper bound of the set $\sup\{x - y : x, y \in S\}$.

Let $\varepsilon > 0$ be arbitrary. Then there exists elements $x_0, y_o \in S$ such that $x_0 > M - \frac{\varepsilon}{2}$ and $y_0 < m + \frac{\varepsilon}{2}$ and so $-y_0 > -m - \frac{\varepsilon}{2}$. Adding this two, we get $x_0 - y_0 > (M - m) - \varepsilon$ and $x_0 - y_0 \in S$. Hence $\sup S - \inf S = M - m = \sup\{x - y : x, y \in S\}$.

Exercise: Let A and B be two non-empty subsets of $\mathbb{R}$ and let $a \leq b$ for all $a \in A$ and $b \in B$. Prove that $\sup A \leq \inf B$.

Solution: Let $b \in B$ be fixed. Then it is to be noted that b is an upper bound of A. Therefore, $\sup A \leq b$. This is true for all $b \in B$. This shows that $\sup A$ is a lower bound of B. Hence $\sup A \leq \inf B$.

Exercise: Let $x, y \in \mathbb{R}$ be such that $x \leq y + \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that $x \leq y$.

Solution: If possible, suppose that $x > y$. Then $x - y > 0$. Therefore, there exists $n \in \mathbb{N}$, such that $n(x - y) > 1$ which implies that $x > y + \frac{1}{n}$ - a contradiction to the hypothesis. Hence we must have $x \leq y$.

Exercise: Let $A, B \in \mathbb{R}$ be bounded above. Find a relation between $\sup(A \cup B)$, $\sup A$ and $\sup B$.

Exercise: Let $A, B \subset \mathbb{R}$ be non-empty. Define

$$A + B = \{a + b : a \in A, b \in B\} = \bigcup_{a \in A}(a + B) = \bigcup_{b \in B}(A + b).$$

Find $A + B$ when a. $A = [-1, 2] = B$ b. $A = B = \mathbb{N}$ c. $A = B = \mathbb{Z}$.

Exercise: Let $M = \sup A$. Let $b \in \mathbb{R}$. Define $b + A = \{b + a : a \in A\}$. Find $\sup(b + A)$.

Solution: By hypothesis, $x \leq M$ for all $x \in A$. Then $b + x \leq b + M$ for all $x \in A$. Therefore, $b + M$ is an upper bound of $b + A$. Let $\varepsilon > 0$ be arbitrary. Then there exists an element $y \in A$ such that

$$y > M - \varepsilon \Longrightarrow b + y > (b + M) - \varepsilon.$$

Hence $\sup(b + A) = b + M = b + \sup A$.

Exercise: Find the supremum of the set $\{1 + \frac{1}{n} : n \in \mathbb{N}\}$.

Solution: Note that $\{1 - \frac{1}{n} : n \in \mathbb{N}\} = \{1\} + \{-\frac{1}{n} : n \in \mathbb{N}\}$.

$\sup\{-\frac{1}{n} : n \in \mathbb{N}\} = \sup(-1)\{\frac{1}{n} : n \in \mathbb{N}\} = (-1)\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$

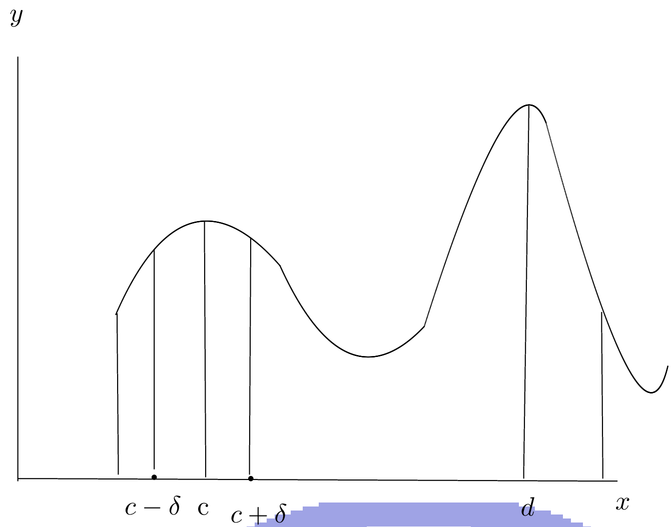
Exercise: Let $M_1 = \sup A$ and $M_2 = \sup B$. Show that $A + B$ is bounded above and that

$$\sup(A + B) = M_1 + M_2.$$

Exercise: Show that $\sup\{1 + \frac{1}{n^2} : n \in \mathbb{N}\} = 2$ and $\inf\{1 - \frac{1}{n^2} : n \in \mathbb{N}\} = 1$.

Exercise: Find $\inf\{x + x^{-1} : x > 0\}$. Is the set bounded above?

Exercise: Find inf and sup of $\{\frac{m+n}{mn} : m, n \in \mathbb{N}\}$.

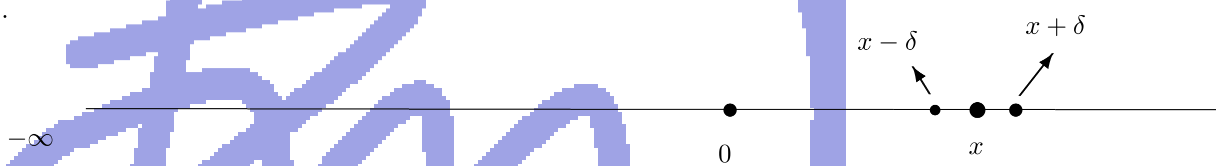


Neighbourhood of a point: Let $c \in \mathbb{R}$. Then the set $(c-\delta, c+\delta) = \{x \in \mathbb{R} : |x-c| < \delta\}$ for some $\delta > 0$ is said to be a neighbourhood (or, nbd.) of the point c. We write it by $N_\delta(c)$. It is clear that $c \in N_\delta(c)$ for all $\delta > 0$. The set $N_\delta(c) - \{c\} = (c-\delta, c+\delta) - \{c\}$ i.e., the set $\{x \in \mathbb{R} : 0 < |x-c| < \delta\}$ is called a deleted δ-nbd. of c.

A set A is a nbd. of a point c if $N_\delta(c) \subset A$ for some $\delta > 0$.

Exercise: If A is a nbd. of a point c and if $A \subset B$, then B is also a nbd. of c.

Exercise: Prove that union and intersection of two nbds. of a point is again a nbd. of that point.



Example:

Consider the real number 1. Then $(1-\frac{1}{2}, 1+\frac{1}{2}) = (\frac{1}{2}, \frac{3}{2})$ is a nbd. of 1.

Consider the real number 0. Then $(-\frac{1}{10}, \frac{1}{10})$ is a nbd. of 0.

Is $(1,2)$ a nbd. of 0? No, since $0 \notin (1,2)$.

Properties of a Neighbourhood:

(1) If A is a nbd. of a point x, then $x \in A$.

(2) If A, B are two nbds of x, then $A \cap B, A \cup B$ are also nbds of x.

Let A, B are two nbds of x. Then there are $\delta_1, \delta_2 > 0$ such that

$$(x-\delta_1, x+\delta_1) \subset A$$

$$(x-\delta_2, x+\delta_2) \subset B$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and $\delta \le \delta_1$ and $\delta \le \delta_2$. Then

$$(x-\delta, x+\delta) \subset (x-\delta_1, x+\delta_1) \subset A$$

Let $y \in (x-\delta, x+\delta)$. Then $|x-y| < \delta \le \delta_1 \implies |x-y| < \delta_1 \implies y \in (x-\delta_1, x+\delta_1)$

$$(x-\delta, x+\delta) \subset (x-\delta_2, x+\delta_2) \subset B$$

which implies that

$$(x-\delta, x+\delta) \subset A \cap B.$$

Hence $A \cap B$ is a nbd. of x.

Open set and Closed set: Let x_0 be a real number and let $\varepsilon > 0$. If a set S contains an ϵ-neighbourhood of x_0, then S is a *neighbourhood* of x_0, and x_0 is an *interior point* of S. The set of interior points of S is the *interior* of S, denoted by S^0. If every point of S is an interior point (that is, $S^0 = S$), then S is **open**. A set S is **closed** if S^c is open.

Let S be a set and $x_0 \in S$. Then x_0 is said to be an interior point of S if there is a $\delta > 0$ such that

$$x_0 \in (x_0 - \delta, x_0 + \delta) \subset S$$

S = four line segments
x_0 is an interior point

$S = (0,1) \cup (5,6) \cup \{10\}$

Set E	Interior Point(s)?
$(-1,1)$	Every point
$[0,1]$	Every ponit except endpoints
$\mathbb{N}$	None
$\mathbb{R}$	Every point
$\mathbb{Q}$	None

Let $S = (a,b)$. Let $x \in S$. Then $a < x < b$. Let $\delta \leq \min\{x-a, b-x\}$. Then $\delta > 0$. Claim:

$$(x-\delta, x+\delta) \subset S$$

Let $y \in (x-\delta, x+\delta)$. Then $y > x - \delta$.

$$S = (0,1) \cup \{2\}$$

$$(0,1)^c = (-\infty, 0] \cup [1, \infty)$$

$$(-\infty, 1]$$

Example: An open interval (a,b) is an open set, because if $x_0 \in (a,b)$ and $\epsilon \leq \min\{x_0 - a, b - x_0\}$, then

$$(x_0 - \epsilon, x_0 + \epsilon) \subset (a,b).$$

The entire line $\mathbb{R} = (-\infty, \infty)$ is open, and therefore $\emptyset (= \mathbb{R}^c)$ is closed. However, $\emptyset$ is also open, for to deny this is to say that $\emptyset$ contains a point that is not an interior point, which is absurd because $\emptyset$ contains no points. Since $\emptyset$ is open, $\mathbb{R} (= \emptyset^c)$ is closed.

Thus, $\mathbb{R}$ and $\emptyset$ are both open and closed. They are the only subsets of $\mathbb{R}$ with this property.

Theorem:

(a) The union of open sets is open.

(b) The intersection of closed sets is closed.

These statements apply to arbitrary collections, finite or infinite, of open and closed sets.

Proof. (a) Let $\mathcal{G}$ be a collection of open sets and

$$S = \cup\{G : G \in \mathcal{G}\}.$$

Let $x_0 \in S$ be arbitrary. Then $x_0 \in G_0$ for some G_0 in $\mathcal{G}$. Since G_0 is open, it is an interior point of G_0. So there is $\varepsilon > 0$ such that

$$x_0 \in (x_0 - \varepsilon, x_0 + \varepsilon) \subset G_0 \subset S$$

i.e., x_0 is an interior point of S and therefore S is open, by definition.

(b) Let us recall DeMorgan's Theorem:

$$\left(\bigcup_i G_i\right)^c = \bigcap_i G_i^c \qquad \text{and} \qquad \left(\bigcap_i F_i\right)^c = \bigcup_i F_i^c$$

Let $\mathcal{F}$ be a collection of closed sets and $T = \cap\{F : F \in \mathcal{F}\}$. Then $T^c = \cup\{F^c : F \in \mathcal{F}\}$ and, since each F^c is open, T^c is open, from (a). Therefore, T is closed, by definition. □

Example: If $-\infty < a < b < \infty$, the set

$$[a, b] = \{x : a \leq x \leq b\}$$

is closed, since its complement is the union of the open sets $(-\infty, a)$ and (b, ∞). We say that $[a, b]$ is a *closed interval*. The set

$$[a, b) = \{x : a \leq x < b\}$$

is a *half-closed* or *half-open interval* if $-\infty < a < b < \infty$, as is

$$(a, b] = \{x : a < x \leq b\};$$

however, neither of these sets is open or closed. (Why not?) *Semi-infinite closed intervals* are sets of the form

$$[a, \infty) = \{x : a \leq x\} \quad \text{and} \quad (-\infty, a] = \{x : x \leq a\},$$

where a is finite. They are closed sets, since their complements are the open intervals $(-\infty, a)$ and (a, ∞), respectively.

This example shows that a set may be both open and closed, and the example shows that a set may be neither. Thus, open and closed are not opposites in this context, as they are in everyday speech.

Example: From Theorem and Example, the union of any collection of open intervals is an open set. (In fact, it can be shown that every nonempty open subset of $\mathbb{R}$ is the union of open intervals.) From Theorem and Example, the intersection of any collection of closed intervals is closed.

It can be shown that the intersection of finitely many open sets is open, and that the union of finitely many closed sets is closed. However, the intersection of infinitely many open sets need not be open, and the union of infinitely many closed sets need not be closed. Let us consider the following example:

Let us consider a family of open sets $\{\left(-\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\}$ in $\mathbb{R}$. Then

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

It is clear that

$$\{0\} \subset \bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Let $x \in \bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then

$$0 \leq |x| \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$.

Now, let $\varepsilon > 0$ be arbitrary. Then there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon$$

for all $n \geq N$. Hence

$$0 \leq |x| \leq \frac{1}{n} < \varepsilon$$

for all $n \geq N$ i.e.,

$$0 \leq |x| < \varepsilon.$$

Then $x = 0$. Hence

$$\bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Definition: Let S be a subset of $\mathbb{R}$. Then

(a) x_0 is a **limit point** of S if every deleted neighbourhood of x_0 contains a point of S i.e., for each $\varepsilon > 0$, the set $(x_0 - \varepsilon, x_0 + \varepsilon) \cap S$ is infinite. Note that a limit point of a set may or may not belongs to the set.

(b) x_0 is a **boundary point** of S if every neighbourhood of x_0 contains at least one point in S and one not in S. The set of boundary points of S is the **boundary** of S, denoted by ∂S. The **closure** of S, denoted by $\overline{S}$, is $\overline{S} = S \cup \partial S$.

(c) x_0 is an **isolated point** of S if $x_0 \in S$ and there is a neighbourhood of x_0 that contains no other point of S. From definition, it follows that an isolated point of a set must belongs to that set.

(d) x_0 is **exterior** to S if x_0 is in the interior of S^c. The collection of such points is the **exterior** of S.

$$\Big((x_0 - \varepsilon, x_0 + \varepsilon) - \{x_0\}\Big) \cap S \neq \phi$$

If a set S admits a limit point, then the set S must be infinite which, in turn, implies that a finite set has no limit point. It is natural to ask, whether an infinite set has a limit point? The answer is not necessarily. Note that the set of all natural numbers $\mathbb{N}$ is an infinite set possessing no limit point.

Bolzano-Weierstarss Theorem: A bounded infinite set has atleast one limit point.

Example: Consider the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. Then by B-W theorem, it has atleast one limit point. It is very easy (not too) to verify that any real number $x \neq 0$ is not a limit point of S.

Let $\varepsilon > 0$ be arbitrary. Consider the neighbourhood $(-\varepsilon, \varepsilon)$ of 0. Now, by Archimedian property, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Also $-\varepsilon < \frac{1}{N}$. Therefore

$$-\varepsilon < \frac{1}{N} < \varepsilon \Longrightarrow \frac{1}{N} \in (-\varepsilon, \varepsilon) \Longrightarrow \frac{1}{n} \in (-\varepsilon, \varepsilon) \forall n \geq N$$

which shows that $(-\varepsilon, \varepsilon) \cap S$ is an infinite set. Hence 0 is a limit point of S.

Example : Let $S = \{sin\frac{n\pi}{2} : n \in \mathbb{N}\} = \{-1, 0, 1\}$.

Consider the sequence $\{sin\frac{n\pi}{2} : n \in \mathbb{N}\} = \{1, 0, -1, 0, 1, 0, -1, \cdots\}$

For all $n \in \mathbb{N}$, the intersection

$$x_n \in \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) \cap S$$

is infinite.

$$0 \leq |x_n - x_0| < \frac{1}{n}$$

Derived set: The set of all limit points of a set S is called the derived set of S and is denoted by S'.

Closure of a set: Let S be a set. Then the closure of the set S is denotes by $\overline{S}$ and is defined by

$$\overline{S} = S \cup S'.$$

Example : Let $S = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$. Then find S'.

Solution: Let $\varepsilon > 0$ be arbitrary.

Case 1: Let m be fixed. Then there is $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon.$$

Also,

$$\frac{1}{m} - \varepsilon < \frac{1}{m} + \frac{1}{N} < \frac{1}{m} + \varepsilon \Longrightarrow \frac{1}{m} + \frac{1}{N} \in \left(\frac{1}{m} - \varepsilon, \frac{1}{m} + \varepsilon\right) \Longrightarrow \frac{1}{m} + \frac{1}{n} \in \left(\frac{1}{m} - \varepsilon, \frac{1}{m} + \varepsilon\right) \forall n \geq N$$

which implies that $\frac{1}{m}$ is a limit point of the set S.

Case 2: None of m, n is fixed. Then there exists $M, N \in \mathbb{N}$ such that

$$\frac{1}{M} < \frac{\varepsilon}{2} \qquad \text{and} \qquad \frac{1}{N} < \frac{\varepsilon}{2}.$$

Then

$$-\varepsilon < \frac{1}{M} + \frac{1}{N} < \varepsilon \Longrightarrow \frac{1}{M} + \frac{1}{N} \in (-\varepsilon, \varepsilon) \Longrightarrow \frac{1}{m} + \frac{1}{n} \in (-\varepsilon, \varepsilon)\, \forall m \geq M, n \geq N$$

which implies that 0 is a limit point of the set S.

Hence $S' = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

Exercise: Find the derived set of the set $S = \{\frac{1}{2^m} + \frac{1}{2^n} : m, n \in \mathbb{N}\}$.

Exercise: Find the derived set of the set $S = (0, 1) \cup \{2\}$.

NOTE: It is to be noted that every point of a set is either an isolated point or a limit point of the set. It is also to be noted that a point x_0 is a limit point of a set S if and only if $G \cap (S \setminus \{x_0\}) \neq \phi$ for each open set G containing x_0. The set ϕ and $\mathbb{R}$ are both open and closed. The term "term" is sometimes called "point of accumulation" or "cluster point".

Set E	Isolated Point(s)?	Limit point(s)?	Boundary Point(s)?
$(-1, 1)$	None	$[-1, 1]$	$\{-1, 1\}$
$[0, 1]$	None	$[0, 1]$	$\{0, 1\}$
$\mathbb{N}$	Every Point	None	$\mathbb{N}$
$\mathbb{R}$	None	Every Point of $\mathbb{R}$	None
$\mathbb{Q}$	None	Every Point of $\mathbb{R}$	$\mathbb{R}$
$(-1, 1) \cup [0, 1]$	None	$[-1, 1]$	$\{-1, 1\}$
$(-1, 1) \setminus \{\frac{1}{2}\}$	None	$[-1, 1]$	$\{-1, \frac{1}{2}, 1\}$
$\{1 - \frac{1}{n} : n \in \mathbb{N}\}$		$\{1\}$	$\{1 - \frac{1}{n} : n \in \mathbb{N}\} \cup \{1\}$

Example: Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Then

(a) The set of limit points of S is $(-\infty, -1] \cup [1, 2]$.

(b) $\partial S = \{-1, 1, 2, 3\}$ and $\overline{S} = (-\infty, -1] \cup [1, 2] \cup \{3\}$.

(c) 3 is the only isolated point of S.

(d) The exterior of S is $(-1, 1) \cup (2, 3) \cup (3, \infty)$.

Example For $n \geq 1$, let

$$I_n = \left[\frac{1}{2n+1}, \frac{1}{2n}\right] \quad \text{and} \quad S = \bigcup_{n=1}^{\infty} I_n.$$

Then

(a) The set of limit points of S is $S \cup \{0\}$.

(b) $\partial S = \{x : x = 0 \text{ or } x = 1/n\ (n \geq 2)\}$ and $\overline{S} = S \cup \{0\}$.

(c) S has no isolated points.

(d) The exterior of S is

$$(-\infty, 0) \cup \left[\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right)\right] \cup \left(\frac{1}{2}, \infty\right).$$

Example: Let S be the set of rational numbers. Since every interval contains a rational number, every real number is a limit point of S; thus, $\overline{S} = \mathbb{R}$. Since every interval also contains an irrational number, every real number is a boundary point of S; thus $\partial S = \mathbb{R}$. The interior and exterior of S are both empty, and S has no isolated points. S is neither open nor closed.

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$	NO	YES	$[-1,1]$	$(-1,1)$	$\{-1,1\}$
$[0,1]$	YES	NO	$[0,1]$	$(0,1)$	$\{0,1\}$
$\mathbb{N}$	YES	NO	$\mathbb{N}$	ϕ	$\mathbb{N}$
$\mathbb{R}$	YES	YES	$\mathbb{R}$	$\mathbb{R}$	ϕ
ϕ	YES	YES	ϕ	ϕ	ϕ
$\mathbb{Q}$	NO	NO	$\mathbb{R}$	ϕ	$\mathbb{R}$
$(-1,1) \cup [0,1]$	NO	NO	$[-1,1]$	$(-1,1)$	$\{-1,1\}$
$(-1,1) \setminus \{\frac{1}{2}\}$	NO	YES	$[-1,1]$	$(-1,1) \setminus \{\frac{1}{2}\}$	$\{-1,\frac{1}{2},1\}$
$E = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$	NO	NO	$E \cup \{1\}$	ϕ	$E \cup \{1\}$

The next theorem says that S is closed if and only if $S = \overline{S}$.

Theorem: A set S is closed if and only if no point of S^c is a limit point of S.

Proof. Suppose that S is closed and $x_0 \in S^c$. Since S^c is open, there is a neighbourhood V_{x_0} of x_0 such that $x_0 \in V_{x_0} \subset S^c$ and therefore $V_{x_0} \cap S = \phi$. Hence, x_0 cannot be a limit point of S.

For the converse, let no point of S^c be a limit point of S. We claim that S is closed. To prove this we show that S^c is open. Let $x_0 \in S^c$. Then x_0 is not a limit point of S. Then x_0 must have a neighbourhood, say V_{x_0}, such that $V_{x_0} \cap S = \phi$ and hence $V_{x_0} \subset S^c$. This shows that x_0 is an interior point of S^c and therefore, S^c is open, and hence S is closed. □

The above Theorem can also be stated as follows:

Corollary: A set is closed if and only if it contains all its limit points S.

Example: The set of integers $\mathbb{Z}$ is closed. Note that

$$\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n, n+1).$$

Since $(n, n+1)$ is open for each $n \in \mathbb{Z}$, it follows that $\mathbb{Z}^c$ is open and hence $\mathbb{Z}$ is a closed set. In a similar way, it can be shown that

$$\mathbb{N}^c = (-\infty, 1) \cup \left(\bigcup_{n \in \mathbb{N}} (n, n+1)\right)$$

which implies that $\mathbb{N}$ is closed.

Note: Above Theorem and Corollary are equivalent. However, we stated the theorem as we did because students sometimes incorrectly conclude from the corollary that a closed set must have limit points. The corollary does not say this. If S has no limit points, then the set of limit points is empty and therefore contained in S. Hence, a set with no limit points is closed according to the corollary, in agreement with Theorem. For example, any finite set is closed.

Some important theorems:

Theorem: Let A and B be two sets of real numbers.

(1) If $A \subset B$ then $A' \subset B'$.
(2) $(A \cup B)' = A' \cup B'$.
(3) $(A \cap B)' \subset A' \cap B'$.

The equality in (3) does not hold, in general. For example, let $A = (1, 2)$ and $B = (2, 3)$.Then $A' = [1, 2]$ and $B' = [2, 3]$ and therefore $A' \cap B' = \{2\}$. Now, $A \cap B = \phi$. So $(A \cap B)' = \phi$. Hence the equality does not hold.

Theorem: Let A and B be two sets of real numbers.

(1) If $A \subset B$ then $\overline{A} \subset \overline{B}$.
(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(3) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

The equality in (3) does not hold, in general. For example, let $A = (1, 2)$ and $B = (2, 3)$.Then $\overline{A} = [1, 2]$ and $\overline{B} = [2, 3]$ and therefore $\overline{A} \cap \overline{B} = \{2\}$. Now, $A \cap B = \phi$. So $\overline{A \cap B} = \phi$. Hence the equality does not hold.

Exercise: If S be any set of real numbers and G be an open subset of $\mathbb{R}$, show that

$$G \cap S \neq \phi \Longrightarrow G \cap S' \neq \phi.$$

Exercise: Let F be a closed subset of $\mathbb{R}$ and let G be an open subset of $\mathbb{R}$. Show that $F - G$ is a closed set and $G - F$ is an open set.

Theorem: A set is open if and only if is a neighbourhood of each of its points.

Proof. Let G be an open set. Let $x \in G$. Then x is an interior point of G and so there is an $\varepsilon > 0$ such that

$$x \in (x - \varepsilon, x + \varepsilon) \subset G.$$

Hence G is a neighbourhood of x.

Conversely, let a set G is a neighbourhood of each of its points. Let $x \in G$. Then there exists $\varepsilon_x > 0$ such that

$$x \in (x - \varepsilon_x, x + \varepsilon_x) \subset G \Longrightarrow \{x\} \subset (x - \varepsilon_x, x + \varepsilon_x) \subset G$$

which implies that

$$\bigcup_{x \in G} \{x\} = G \subset \bigcup_{x \in G} (x - \varepsilon_x, x + \varepsilon_x) \subset G \Longrightarrow G = \bigcup_{x \in G} (x - \varepsilon_x, x + \varepsilon_x).$$

This shows that G is an union of open sets and hence is an open set. □

Interior of a set: Let S be a set of real numbers. Then the set of all interior points of S is denoted by S° and is called interior of S.

Lets verify the following properties:

Theorem: Let A and B be two subsets of real numbers. Following statements hold good:

(a) If $A \subset B$, then $A^\circ \subset B^\circ$.
(b) $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.

(c) $A^\circ \cap B^\circ = (A \cap B)^\circ$.

Proof. Left as an exercise. □

Note: It is to be noted that equality in (b) need not hold in general. For example, let $A = [0,1]$, $B = [1,2]$. Then $A^\circ = (0,1), B^\circ = (1,2)$. Again $A \cup B = [0,2]$ and hence $(A \cup B)^\circ = (0,2)$. It then follows that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.

Theorem: The interior of a set is an open set.

Proof. If $S^\circ = \phi$, then there is nothing to proof.

So, let us suppose that $S^\circ \neq \phi$. Let $x \in S^\circ$. Then x is an interior point of S and therefore there exists an $\varepsilon > 0$ such that $x \in (x-\varepsilon, x+\varepsilon) \subset S$. Since $(x-\varepsilon, x+\varepsilon)$ is an open set, it is a neighbourhood of each of its points. Since $(x-\varepsilon, x+\varepsilon) \subset S$, it follows that $(x-\varepsilon, x+\varepsilon) \subset S^\circ$. This shows that x is an interior point of S°. Hence S° is an open set. □

Theorem: The interior of a set S is the largest open subset of S.

Proof. We have already shown that S° is an open set. Let G be an open subset of S. Claim: $G \subset S^\circ$.

Let $x \in G$. Then x is an interior point of G. Then there is an $\varepsilon > 0$ such that $x \in (x-\varepsilon, x+\varepsilon) \subset G$. Since $G \subset S$, it follows that $x \in (x-\varepsilon, x+\varepsilon) \subset S \implies x \in S^\circ$. Hence $G \subset S^\circ$. □

Theorem: The derived set S' and $\overline{S}$ are closed sets.

Proof. To prove that S' is a closed set, we must show that S' contains all of its limit points. Let x be a limit point of S'. Let $\varepsilon > 0$ be arbitrary. Then $(x-\varepsilon, x+\varepsilon) \cap S'$ is an infinite set. Let $y \in (x-\varepsilon, x+\varepsilon) \cap S'$ such that $y \neq x$. Since $(x-\varepsilon, x+\varepsilon)$ is an open set, it is a neighbourhood of y. So there is an $\varepsilon_1 > 0$ such that $(y-\varepsilon_1, y+\varepsilon_1) \subset (x-\varepsilon, x+\varepsilon)$. Since y is a limit point of S, the set $(y-\varepsilon_1, y+\varepsilon_1) \cap S$ is infinite. It follows that the set $(x-\varepsilon, x+\varepsilon) \cap S$ is an infinite set as well. This implies that x is a limit point of S, which means that $x \in S'$. Since S' contains all of its limit points, it is a closed set. □

A set S is closed iff $S = \overline{S}$. It is to be noted that $S \subset \overline{S}$. Also, if S is closed then

$$S' \subset S \implies S \cup S' \subset S \cup S \implies \overline{S} \subset S$$

Now let $S = \overline{S} = S \cup S'$. This implies that $S' \subset S$ and hence S is closed.

A set S is open iff $S = S^\circ$.

Exercise: Describe the following sets as open, closed, or neither, and find S^0, $(S^c)^0$, and $(S^0)^c$.

(a) $S = (-1, 2) \cup [3, \infty)$.
(b) $S = (-\infty, 1) \cup (2, \infty)$.
(c) $S = [-3, -2] \cup [7, 8]$.
(d) $S = \mathbb{Z}$.

Exercise:

(a) Show that the intersection of finitely many open sets is open.
(b) Give an example showing that the intersection of infinitely many open sets may fail to be open.

Exercise:

(a) Show that the union of finitely many closed sets is closed.
(b) Give an example showing that the union of infinitely many closed sets may fail to be closed.

Exercise: Find the set of limit points of S, ∂S, $\overline{S}$, the set of isolated points of S, and the exterior of S.

(a) $S = (-\infty, -2) \cup (2, 3) \cup \{4\} \cup (7, \infty)$.
(b) $S = \mathbb{Z}$.
(c) $S = \cup\{(n, n+1) : n \in \mathbb{Z}\}$.
(d) $S = \{x : x = 1/n, n = 1, 2, 3, \ldots\}$.

Exercise:

(a) Prove: A limit point of a set S is either an interior point or a boundary point of S.
(b) Prove: An isolated point of S is a boundary point of S^c.
(c) A boundary point of a set S is either a limit point or an isolated point of S.
(d) A set S is closed if and only if $S = \overline{S}$.
(e) Prove or disprove: A set has no limit points if and only if each of its points is isolated.
(f) Prove: If S is bounded above and $\beta = \sup S$, then $\beta \in \partial S$. State the analogous result for a set bounded below.
(g) Prove: If S is closed and bounded, then $\inf S$ and $\sup S$ are both in S.
(h) If a nonempty subset S of $\mathbb{R}$ is both open and closed, then $S = \mathbb{R}$.

Exercise: Let S be an arbitrary set. Prove:

(a) ∂S is closed.
(b) S^0 is open.
(c) The exterior of S is open.
(d) The limit points of S form a closed set.
(e) $\overline{(\overline{S})} = \overline{S}$.

Exercise: Give counterexamples to the following false statements.

(a) The isolated points of a set form a closed set.
(b) Every open set contains at least two points.
(c) If S_1 and S_2 are arbitrary sets, then $\partial(S_1 \cup S_2) = \partial S_1 \cup \partial S_2$.
(d) If S_1 and S_2 are arbitrary sets, then $\partial(S_1 \cap S_2) = \partial S_1 \cap \partial S_2$.
(e) The supremum of a bounded nonempty set is the greatest of its limit points.
(f) If S is any set, then $\partial(\partial S) = \partial S$.
(g) If S is any set, then $\partial \overline{S} = \partial S$.
(h) If S_1 and S_2 are arbitrary sets, then $(S_1 \cup S_2)^0 = S_1^0 \cup S_2^0$.

Open Coverings: A collection $\mathcal{G}$ of open sets is an **open covering** of a set S if every point in S is contained in a set G belonging to $\mathcal{G}$; that is, if $S \subset \cup\{G : G \in \mathcal{G}\}$. The open cover $\mathcal{G}$ has a finite **subcover** if S is contained in the union of a finite number of sets in$\mathcal{G}$.

In other words, a collection $\mathcal{G}$ of open sets is an open cover of S if $S \subset \bigcup_{G \in \mathcal{G}} G$. The open cover $\mathcal{G}$ has a finite subcover if there exists $G_1, G_2, \cdots, G_n$ in $\mathcal{G}$ such that $S \subset \bigcup_{i=1}^{n} G_i$. To say another way, $\mathcal{G}_0$ is a finite subcover if $\mathcal{G}_0 \subset \mathcal{G}$, the set $\mathcal{G}_0$ is finite and $S \subset \bigcup_{G \in \mathcal{G}_0} G$.

Example: The sets

$$S_1 = [0, 1],\ S_2 = \{1, 2, \ldots, n, \ldots\},$$
$$S_3 = \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}, \quad \text{and} \quad S_4 = (0, 1)$$

are covered by the families of open intervals

$$\mathcal{H}_1 = \left\{ \left(x - \frac{1}{N}, x + \frac{1}{N} \right) \Big| \, 0 < x < 1 \right\}, \quad (\mathbb{N} = \text{positive integer}),$$
$$\mathcal{H}_2 = \left\{ \left(n - \frac{1}{4}, n + \frac{1}{4} \right) \Big| \, n = 1, 2, \dots \right\},$$
$$\mathcal{H}_3 = \left\{ \left(\frac{1}{n + \frac{1}{2}}, \frac{1}{n - \frac{1}{2}} \right) \Big| \, n = 1, 2, \dots \right\},$$

and

$$\mathcal{H}_4 = \{(0, \rho) | \, 0 < \rho < 1\},$$

respectively.

There are many possible open covers for any given set of real numbers. For the interval $(0, 1)$, all the following collections are open covers

$\mathcal{G}_1 = \left\{ \left(\frac{1}{n}, 1 \right) : n \in \mathbb{N} \setminus \{1\} \right\}$.
$\mathcal{G}_2 = \left\{ \left(\frac{1}{n}, n \right) : n \in \mathbb{N} \setminus \{1\} \right\}$.
$\mathcal{G}_3 = \left\{ \left(\frac{1}{2n}, 1 - \frac{1}{2n} \right) : n \in \mathbb{N} \setminus \{1\} \right\}$.
$\mathcal{G}_4 = \left\{ \left(\frac{n}{4}, \frac{n+2}{4} \right) : n \in \mathbb{Z} \right\}$.
$\mathcal{G}_5 = \left\{ \left(\frac{1}{r}, r \right) : r > 1 \right\}$.
$\mathcal{G}_6 = \{(-r, 1 - r) : 0 < r < 1\}$.
$\mathcal{G}_7 = \{(-0.2, 0.3), (0.1, 0.4), (0.3, 0.9), (0.7, 1.4)\}$.
$\mathcal{G}_8 = \{(0, 1)\}$.

It is important to verify that $\mathcal{G}_2 \subset \mathcal{G}_5$, but $\mathcal{G}_2$ is countably infinite subcover. The open cover $\mathcal{G}_4$ contains a finite subcover, say, $\{(0, 0.5), (0.25, 0.75), (0.5, 1.0)\}$.

Compact set: A set S of real numbers is said to be compact if every open cover of S has a finite subcover.

Using only definition, it is much easier to prove that a set is not compact that to prove it compact. Note that a set is non compact, if one can find an open cover that does not have a finite subcover. For example, the set $\mathbb{N}$ is not compact since the open cover $\{(n - 1, n + 1) : n \in \mathbb{N}\}$ has no finite subcover. Not only that, it is to be noted that if any open set is removed from this collection, then the new collection fails to cover $\mathbb{N}$.

Example: The set $\mathbb{R}$ is not compact since the open cover $\{(-n, n) : n \in \mathbb{N}\}$ admits no finite subcover. This follows from the fact that union of finite number of members of this cover is a bounded set.

Example: $(0, 1)$ is not compact: The open cover $\mathcal{G}_1$ of $(0, 1)$ has no finite subcover. If possible suppose that, this statement is not true. Then there is a finite collection, say, $\left\{ \left(\frac{1}{n_i}, 1 \right) : 1 \le i \le k \right\}$ of $\mathcal{G}_1$ which covers $(0, 1)$. Let $m = \max\{n_i : 1 \le i \le k\}$. Then

$$\bigcup_{i=1}^{k} \left(\frac{1}{n_i}, 1 \right) = \left(\frac{1}{m}, 1 \right).$$

It follows that no finite subset of $\mathcal{G}_1$ can cover $(0, 1)$. Hence $(0, 1)$ is not compact.

Let us now take an example of a compact set.

Example: Show that the set $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ is compact.

Answer: Let $\mathcal{G}$ be an open cover of S. Then there is $G_0 \in \mathcal{G}$ such that $0 \in G_0$. Since G_0 is an open set, there is an $r > 0$ such that $(-r, r) \subset G_0$We can choose a positive integer N such that $\frac{1}{n} < r$ for all $n > N$. Since $\mathcal{G}$ is an open cover of S, there exists $G_1, G_2, \cdots, G_N$ in $\mathcal{G}$ such that $\frac{1}{n} \in G_n$ for all $n = 1, 2, \cdots, N$. It then follows that $\{G_0, G_1, G_2, \cdots, G_N\}$ is a finite subcover of S. Since the open cover $\mathcal{G}$ was taken arbitrarily, it follows that every open cover of S has a finite subcover. Hence the set S is compact.

Theorem: A closed subset of a compact set is compact.

Proof. Let S be a compact set and let E be a closed subset of S. Let $\mathcal{G}$ be an open cover of E. Then the collection $\mathcal{G}\bigcup\{E^c\}$ is an open cover of S. Since S is compact, this collection has a finite subcover $\mathcal{G}_1$ of S. Now, the collection $\mathcal{G}_1 \setminus \{E^c\}$ is a finite subcover of $\mathcal{G}$. Since $\mathcal{G}$ was taken arbitrarily, it follows that every open cover of E has a finite subcover. Hence E is compact. □

Heine-Borel Theorem: Every closed bounded set of real numbers is compact i.e., if $\mathcal{G}$ is an open covering of a closed and bounded subset S of the real line, then S has an open covering $\widetilde{\mathcal{G}}$ consisting of finitely many open sets belonging to $\mathcal{G}$.

Proof. We first show that every closed bounded interval $[a, b]$ is compact. Let $\mathcal{G}$ be an open cover of $[a, b]$. Let

$$E = \{x \in [a, b] : [a, x] \text{ can be covered by a finite number of elements of } \mathcal{G}\}.$$

It is clear that E is non-empty, since $a \in E$. Also E is bounded above by b. Hence by completeness property, $\sup E$ exists and let us call it z. We aim to show that $z \in E$ and $z = b$.

Since $\mathcal{G}$ is an open cover of $[a, b]$, there is $G_z \in \mathcal{G}$ such that $z \in G_z$. Since G_z is an open set, there is an $\varepsilon > 0$ such that $z \in (z-\varepsilon, z+\varepsilon) \subset G_z$. Again, since $z = \sup E$, there is an element $c \in E$ such that $z - \varepsilon < c \le z < z + \varepsilon$. Since $c \in E$, the interval $[a, c]$ can be covered by a finite number of elements of $\mathcal{G}$. Now $[a, z] = [a, c] \cup [c, z]$ and $[c, z] \subset (z-\varepsilon, z+\varepsilon) \subset G_z$. Hence $[a, z]$ can be covered by a finite number of elements of $\mathcal{G}$. Consequently, $z \in E$.

To show that $z = b$, we assume that $z < b$. Since $z < b$ and G_z is an open set containing z, we choose a point d such that $z < d < z + \varepsilon < b$. Now, $[a, d] = [a, z] \cup [z, d]$. Note that $[a, z]$ can be covered by a finite number of elements of $\mathcal{G}$. Also $[z, d] \subset (z-\varepsilon, z+\varepsilon) \subset G_z$. Hence $[a, d]$ can be covered by a finite number of elements of $\mathcal{G}$ which implies that $d \in E$-a contradiction. Therefore we must have $z = b$ and consequently, $[a, b]$ can be covered by a finite number of elements of $\mathcal{G}$ and hence $[a, b]$ is compact.

Now let S be a closed bounded subset of real numbers. So there exists a real number $M > 0$ such that $K \subset [-M, M]$. By the first part, $[-M, M]$ is compact. Since S is a closed subset of a compact set $[-M, M]$, it follows that S is compact. Hence the proof is complete. □

Henceforth, we will say that a closed and bounded set is **compact**. The Heine–Borel theorem says that any open covering of a compact set S contains a finite collection that also covers S. This theorem and its converse show that we could just as well define a set S of reals to be compact if it has the Heine–Borel property; that is, if every open covering of S contains a finite subcovering. The same is true of $\mathbb{R}^n$. This definition generalizes to more abstract spaces (called **topological spaces**) for which the concept of boundedness need not be defined.

The Bolzano–Weierstrass Theorem: As an application of the Heine–Borel theorem, we prove the following theorem of Bolzano and Weierstrass.

Theorem: Every bounded infinite set of real numbers has at least one limit point.

Proof. We will show that a bounded nonempty set without a limit point can contain only a finite number of points. If S has no limit points, then S is closed and every point x of S has an open neighborhood N_x that contains no point of S other than x. The collection

$$\mathcal{H} = \{N_x : x \in S\}$$

is an open covering for S. Since S is also bounded, Heine-Borel theorem implies that S can be covered by a finite collection of sets from $\mathcal{H}$, say $N_{x_1}, \ldots, N_{x_n}$. Since these sets contain only $x_1, \ldots, x_n$ from S, it follows that $S = \{x_1, \ldots, x_n\}$. $\square$

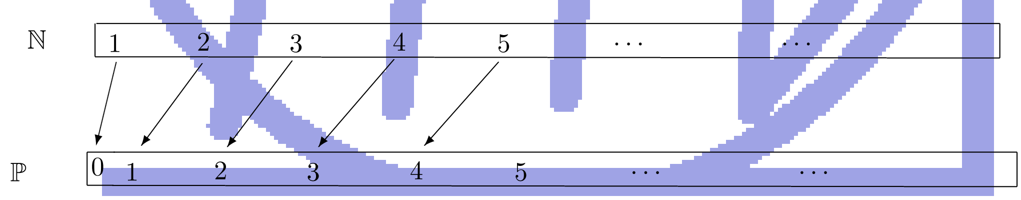
Exercise: Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of real numbers such that $x_n \to x_0$. Prove that $\{x_n : n = 0, 1, 2, \cdots\}$ is compact.

Idea of countable sets, uncountable sets and uncountability of $\mathbb{R}$: Are there more rational numbers than there are integers? How about real numbers; are there more of them than there are of rationals? Are there fewer numbers in the interval $(0, 1)$ than in $(0, 2)$ or than in $\mathbb{R}$ itself? Mathematicians would say that the answers to these questions are no, yes, and no, respectively, but what do these answers-and the questions-mean? Read on, and leave your intuition behind; it won't help much here.

Sets that we can match up with $\{1, 2, \cdots, n\}$ or with $\mathbb{N} = \{1, 2, \cdots\}$ are especially important for both theoretical and practical reasons. The ones that match up with $\{1, 2, \cdots, n\}$ are the finite sets of size n. Sets that are the same size as $\mathbb{N}$ are called **countably infinite (or enumerable)**. Thus a set S is **countably infinite** if and only if there exists a one-to-one correspondence $\mathbb{N}$ onto S i.e., if there is a bijection between the sets $\mathbb{N}$ and S. A set is **countable** if it is finite or is countably infinite. One is able to count or list such a nonempty set by matching it with $\{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$, or with the whole set $\mathbb{N}$. In the infinite case, the list will never end. As one would expect, a set is **uncountable** if it is not countable.

Example: (a) The set S of all even positive integers are countable, since the mapping $f(n) = 2n$ for all $n \in \mathbb{N}$ is a bijection from $\mathbb{N}$ onto S

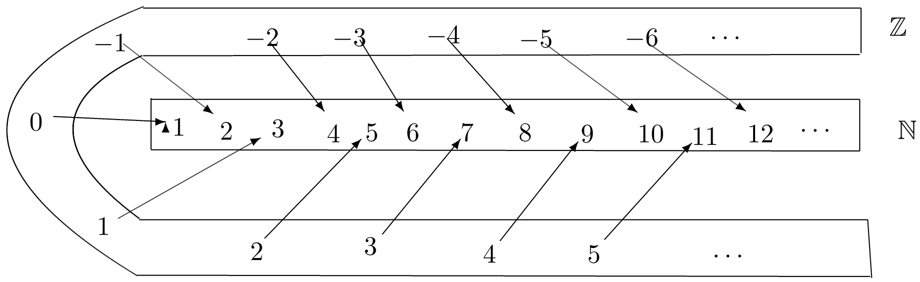
(b) The set $\mathbb{P} = \{0, 1, 2, \cdots\}$ is countably infinite because $f(n) = n-1$ defines a one-to-one function f mapping $\mathbb{N}$ onto $\mathbb{P}$. Its inverse f is a one-to-one mapping of $\mathbb{P}$ onto $\mathbb{N}$; note that $f^{-1}(n) = n + 1$ for $n \in \mathbb{N}$. Even though $\mathbb{N}$ is a proper subset of $\mathbb{P}$, by our definition $\mathbb{N}$ is the same size as $\mathbb{P}$. This may be surprising, since a similar situation does not occur for finite sets. Oh well, $\mathbb{P}$ has only one element that is not in $\mathbb{N}$.



(c) The set $\mathbb{Z}$ of all integers is also countably infinite. The Figure given below shows a one-to-one function f from $\mathbb{Z}$ onto $\mathbb{N}$. We have found it convenient to bend the picture of $\mathbb{Z}$. This function can be given by a formula, if desired:

$$f(n) = \begin{cases} 2n+1 & \text{for } \ n \geq 0 \\ -2n & \text{for } n < 0 \end{cases}$$

Even though $\mathbb{Z}$ looks about twice as big as $\mathbb{N}$, these sets are of the same size. Beware! For infinite sets, your intuition may be unreliable. Or, to take a more positive approach, you may need to refine your intuition when dealing with infinite sets.



There are sets that are uncountable, i.e., not the same size as $\mathbb{N}$. Let us take the following example:

Example: The interval $[0,1)$ is uncountable. If it were countable, there would exist a one-to-one function f mapping $\mathbb{N}$ onto $[0,1)$. We will show that this is impossible. Each number in $[0,1)$ has a decimal expansion $0.d_l d_2 d_3 \cdots$, where each d_j is a digit in $\{0,1,2,3,4,5,6,7,8,9\}$. In particular, each number $f(k)$ has the form $0.d_{1k}d_{2k}d_{3k}\cdots$; here d_{nk} represents the nth digit in $f(k)$.

$$\boxed{\begin{array}{l} f(1) = 0.\mathbf{d_{11}}d_{21}d_{31}d_{41}\cdots \\ f(2) = 0.d_{12}\mathbf{d_{22}}d_{32}d_{42}\cdots \\ f(3) = 0.d_{13}d_{23}\mathbf{d_{33}}d_{43}\cdots \\ f(4) = 0.d_{14}d_{24}d_{34}\mathbf{d_{44}}\cdots \end{array}}$$

Consider the Figure shown above and look at the indicated diagonal digits $d_{11}, d_{22}, d_{33}, \cdots$. We now define the sequence d^*, whose nth term d_n^* is constructed as follows: if $d_{nn} \neq 1$, let $d_n^* = 1$, and if $d_{nn} = 1$, let $d_n^* = 2$ i.e.,

$$d_n^* = \begin{cases} 1 & \text{if } d_{nn} \neq 1 \\ 2 & \text{if } d_{nn} = 1 \end{cases}$$

The point is that $d_n^* \neq d_{nn}$ for all $n \in \mathbb{N}$. Now $0.d_1^*d_2^*d_3^*\cdots$ is a decimal expansion for a number a in $[0,1)$ that is different from $f(n)$ in the nth digit for each $n \in \mathbb{N}$. Thus a cannot be one of the numbers $f(n)$; i.e., a is not in $Im(f)$, so f does not map $\mathbb{N}$ onto $[0,1)$. Thus $[0,1)$ is uncountable.

The proof can be modified to prove that $\mathbb{R}$ and $(0,1)$ are uncountable; in fact, all intervals $[a,b], [a,b), (a,b]$, and (a,b) are uncountable for $a < b$. In view of Exercise 9, another way to show that these sets are uncountable is to show that they are in one-to-one correspondence with each other. In fact, they are also in one-to-one correspondence with unbounded intervals. Showing the existence of such one-to-one correspondences can be challenging. We provide a couple of the trickier arguments in the next example and ask for some easier ones in Exercise 3.

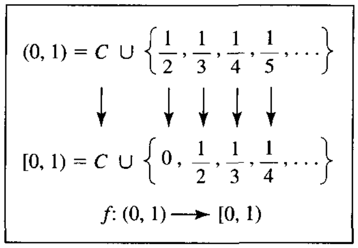
Example: (a) It is easy to show that $(0,1)$ and $(0,2)$ are the same size; the function f defined by $f(x) = 2x$ gives a one-to-one correspondence from $(0,1)$ onto $(0,2)$. More generally, the linear function $f(x) = ax + b$ with $a > 0$ maps $(0,1)$ one-to-one onto $(b, a+b)$.

(b) We can show that $[0,1)$ and $(0,1)$ have the same size. No simple formula provides us with a one-to-one mapping between these sets. The trick is to isolate some infinite sequence in $(0,1)$, say $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ and then map this sequence onto $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, while leaving the complement fixed. That is, let

$$C = (0,1) \setminus \left\{ \frac{1}{n} : n = 1, 2, 3, \cdots \right\}$$

and define

$$f(x) = \begin{cases} 0 & \text{if } \ x = \frac{1}{2} \\ \frac{1}{n-1} & \text{if } x = \frac{1}{n} \text{ for some integer } n \geq 3 \\ x & \text{if } x \in C \end{cases}$$



Theorem: (a) An infinite set contains an enumerable set.

(b) Subsets of countable sets are countable.

(c) The union of countably many countable sets is countable.

Proof. (a) Let A be an infinite set. Let us define a mapping $f : \mathbb{N} \to A$ by

$f(1)$ = an element from A, say x_1

$f(2)$ = an element from $A \setminus \{x_1\}$, say x_2

$f(3)$ = an element from $A \setminus \{x_1, x_2\}$, say x_3

$\cdots$

$f(n)$ = an element from $A \setminus \{x_1, x_2, \cdots, x_{n-1}\}$, say x_n

$\cdots$.

It is very important to note that f is an one-to-one mapping of $\mathbb{N}$ into A. Let us take $B = f(\mathbb{N}) = \{x_1, x_2, \cdots, x_n, \cdots\}$. Then $f : \mathbb{N} \to B$ is a bijection and therefore the subset B of A is countable.

(b) It is enough to show that subsets of $\mathbb{N}$ are countable. Consider a subset A of $\mathbb{N}$. Clearly, A is countable if A is finite. Suppose that A is infinite. Define $f(1)$ to be the least element in A. Then define $f(2)$ to be the least element in $A \setminus \{f(1)\}$, $f(3)$ to be the least element in $A \setminus \{f(1), f(2)\}$, etc. Continue this process so that $f(n + 1)$ is the least element in the non-empty set $A \setminus \{f(k) : 1 \leq k \leq n\}$ for each $n \in \mathbb{N}$. It is easy to verify that this recursive definition provides a one-to-one function f mapping $\mathbb{N}$ onto A, so A is countable.

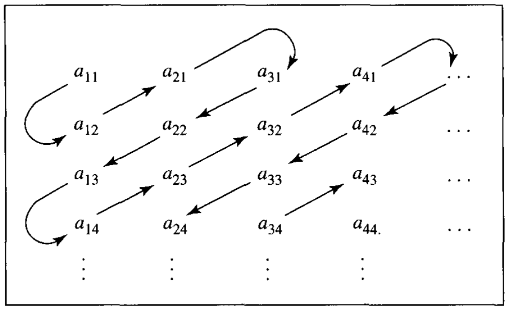
(c) The statement in part (c) means that, if I is a countable set and if $\{A_i : i \in I\}$ is a family of countable sets, then the union $\cup A_i$ is countable. We may assume that each A_i is non-empty and that $\cup A_i$ is infinite, and we may assume that $I = \mathbb{N}$ or that I has the form $\{1, 2, \cdots, n\}$. If $I = \{1, 2, \cdots, n\}$, we can define $A_i = A_n$ for $i > n$ and obtain a family $\{A_i : i \in \mathbb{N}\}$ with the same union. Thus we may assume that $I = \mathbb{N}$. Each set A_i is finite or countably infinite. By repeating elements if A_i is finite, we can list each A_i as follows:

$$A_i = \{a_{1i}, a_{2i}, a_{3i}, a_{4i}, \cdots\}.$$

The elements in $\cup A_i$ can be listed in an array as in Figure given below. The arrows in the figure suggest a single listing for $\cup A_i$:

$$a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, \cdots . \qquad (*)$$

Some elements may be repeated, but the list includes infinitely many distinct elements, since $\cup A_i$ is infinite. Now a one-to-one mapping f of $\mathbb{N}$ onto $\cup A_i$ is obtained as follows: $f(1) = a_{11}$, $f(2)$ is the next element listed in $(*)$ different from $f(1)$, $f(3)$ is the next element listed in $(*)$ different from $f(1)$ and $f(2)$, etc.



Thus we get a one-to-one mapping from $\mathbb{N}$ onto $\cup A_i$ which shows that $\cup A_i$ is countable. $\square$

Example: We use the above theorem to show that the set $\mathbb{Q}$ of all rational numbers is countable. To show this we first show that the set $\mathbb{Q}^+$ of all positive rational numbers is countable. Now for each n in $\mathbb{N}$, let

$$\begin{aligned}
A_1 &= \left\{\tfrac{m}{1} : m \in \mathbb{N}\right\} = \left\{\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{3}{1}, \cdots, \tfrac{n}{1}\right\}.\\
A_2 &= \left\{\tfrac{m}{2} : m \in \mathbb{N}\right\} = \left\{\tfrac{1}{2}, \tfrac{2}{2}, \tfrac{3}{2}, \cdots, \tfrac{n}{2}\right\}.\\
A_3 &= \left\{\tfrac{m}{3} : m \in \mathbb{N}\right\} = \left\{\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \cdots, \tfrac{n}{3}\right\}.\\
&\cdots.\\
A_k &= \left\{\tfrac{m}{k} : m \in \mathbb{N}\right\} = \left\{\tfrac{1}{k}, \tfrac{2}{k}, \tfrac{3}{k}, \cdots, \tfrac{n}{k}\right\}.\\
&\cdots.
\end{aligned}$$

Since A_n is countable, by the above theorem, it follows that $\bigcup\limits_{n=1}^{\infty} A_n$ is countable i.e., the set of all positive rational numbers is countable.

Now it is obvious that the set $\mathbb{Q}^-$ of all negative rational numbers is countable. It is clear from the fact that the mapping $f : \mathbb{Q}^+ \to \mathbb{Q}^-$ defined by $f(x) = -x$ for all $x \in \mathbb{Q}$ is a bijection. Finally it follows that

$$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$$

is also countable.

Example: The set $\mathbb{N} \times \mathbb{N}$ is counatble.

Solution: Let us define a mapping $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by $f(m,n) = 2^m 3^n$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. It is clear that f is injective, since

$$\begin{aligned}
f(m,n) = f(p,q) &\implies 2^m 3^n = 2^p 3^q\\
&\implies 2^{m-p} = 3^{q-n}\\
&\implies m-p = 0, q-n = 0\\
&\implies m = p, n = q\\
&\implies (m,n) = (p,q).
\end{aligned}$$

Note that $f(\mathbb{N}\times\mathbb{N})$ is a proper subset of $\mathbb{N}$ since the elements $5, 7, 10, \cdots$ are not in $f(\mathbb{N}\times\mathbb{N})$. Hence $f : \mathbb{N}\times\mathbb{N} \to f(\mathbb{N}\times\mathbb{N})$ is a bijective mapping. Since subset of a countable set is countable, the set $f(\mathbb{N}\times\mathbb{N})$ is countable. Hence $\mathbb{N}\times\mathbb{N}$ is countable.

Exercise: Give one-to-one correspondences between the following pairs of sets:

(a) $(0,1)$ and $(-1,1)$ (b) $[0,1)$ and $(0,1]$
(c) $[0,1]$ and $[-5,8]$ (d) $(0,1)$ and $(1,\infty)$
(e) $(0,1)$ and $(0,\infty)$ (f) $\mathbb{R}$ and $(0,\infty)$.

Exercise: Let $E = \{n \in \mathbb{N} : n \text{ is even}\}$. Show that E and $\mathbb{N}\setminus E$ are countable by exhibiting one-to-one correspondences $f : \mathbb{N} \to E$ and $g : \mathbb{N} \to \mathbb{N}\setminus E$.

Exercise: Show that if there is a one-to-one correspondence of a set S onto some countable set, then S itself is countable.

Exercise: Prove that if f maps S onto T and S is countable, then T is countable.