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Tracing of curves

For the given equation of a curve, though the nature of the curve can be studied by investing properties such as continuity and differentiability at different points of the curve, a diagrammatic representation of the curve often helps the readers to understand these properties in a better way. In this chapter, we first discuss the general rules to be followed to trace the curve of a given equation, given in Cartesian, parametric and polar form.

(a) Tracing of curves in cartesian form

Let $y = f(x)$ or $f(x, y) = 0$ be the equation of a given curve in Cartesian form.

Symmetry of a curve about x-axis: Now, if the curve be symmetric about x-axis, then both the points (x, y) and $(x, -y)$ lie on the curve (shown in Fig. 1), i.e., $f(x, y) = 0$ and $f(x, -y) = 0$, and hence, we may write $f(x, -y) = \pm f(x, y)$. In other words, if by replacing y by $-y$, the curve remains unchanged, then the curve is symmetrical about the x-axis.

Symmetry of a curve about y-axis: If the curve be symmetric about y-axis, then both the points (x, y) and $(-x, y)$ lie on the curve (shown in Fig. 1), i.e., $f(x, y) = 0$ and $f(-x, y) = 0$, and hence, we may write $f(-x, y) = \pm f(x, y)$. In other words, if by replacing x by $-x$, the curve remains unchanged, then the curve is symmetrical about the y-axis.

Explanation: Let us consider a curve as shown in the figure. Now, if (x, y) be any point on the curve, then the image of the point with respect to x-axis is $(x, -y)$. If the curve is symmetric, then both (x, y) and $(x, -y)$ will lie on the curve, i.e., both $f(x, y) = 0$ and $f(x, -y) = 0$, and therefore, we must have $f(x, -y) = \pm f(x, y)$.

Similarly, if by replacing x by $-x$ in the equation of the curve, the equation of curve remains unchanged, i.e., if $f(-x, y) = \pm f(x, y)$, then the curve is symmetrical about y-axis.

Note. Sometimes, we need to find whether a curve is symmetrical about the line $y = x$. If by interchanging x and y in the equation of the curve, the equation of curve remains unchanged, i.e., if $f(y, x) = f(x, y)$, then the curve is symmetrical about the line $y = x$.

Some examples:

Example 1. Consider the curve $x^2 + y^2 = a^2$, which is a circle with centre at $(0, 0)$ and radius a units. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. In a similar manner, since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the y-axis. This is shown in Fig. 2.

Example 2. Consider the curve $(x - a)^2 + y^2 = r^2$, which is a circle with centre at $(a, 0)$ (i.e., on x-axis) and radius r units. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. In a similar manner, since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the y-axis. This is shown in Fig. 3.

Example 3. Consider the curve $y^2 = 4ax$, which is a parabola with vertex at $(0, 0)$ and the x-axis being the axis. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. However, by replacing x by $-x$, the curve does not remain unchanged, so the curve is not symmetrical about the y-axis. This is shown in Fig. 4.

Example 4. Consider the curve $x^{2/3} + y^{2/3} = a^{2/3}$, an Astroid, shown in Fig. 4. Since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the x-axis. In a similar manner, since by replacing y by $-y$, the curve remains unchanged, the curve is symmetrical about the y-axis.

(b) Rules to trace curves given in cartesian form

To trace curves in the cartesian form, say, $f(x, y) = 0$, we note the following points:

(i) **Symmetrical about the co-ordinate axes:** Check whether the given curve is symmetrical about the co-ordinate axes.

(ii) **Obtain the points of intersection with x-axis and y-axis:** To find the points of intersection of the curve with x-axis, we put $y = 0$ in the equation of the curve. Similarly, to find the points of intersection of the curve with y-axis, we put $x = 0$ in the equation of the curve.

(iii) **Obtain the boundary of the curve:** If by putting $x < -a$ or $x > b$ in the equation of the curve, y^2 becomes negative (i.e., y takes imaginary values), that means no part of the curve lies to the left of the line $x = -a$ or to the right of the line $x = b$. Similarly, if by putting $y < -c$ or $y > d$ in the equation of the curve, x^2 becomes negative (i.e., x takes imaginary values), that means no part of the curve lies below the line $y = -c$ or above the line $y = d$.

(iv) **Tangent of curve at origin:** If the curve passes through the origin and the equation of the curve is given by a polynomial equation, the equation of the tangent exists and is obtained by equating to zero, the lowest degree terms of the equation.

(v) **Asymptotes to the curve parallel to the co-ordinate axes:** Let the equation of the curve be given by a polynomial equation. Obtain the co-efficient of the highest degree term of x. If the co-efficient is a non-constant, i.e., a function of y, say, $\phi(y)$, then the asymptotes parallel to x-axis exists, and are given by $\phi(y) = 0$. Obtain the co-efficient of the highest degree term of y. If the co-efficient is a non-constant, i.e., a function of x, say, $\psi(x)$, then the asymptotes parallel to y-axis exists, and are given by $\psi(x) = 0$.

Tracing of curves in polar form

For a curve given in polar form $r = f(\theta)$ (or, $f(r,\theta) = 0$), if the equation of the curve remains unchanged by replacing θ by $-\theta$, then the curve is symmetrical about the initial line. If the curve is symmetrical about the initial line, obtain the values of the radius vector r for different values of θ in the range 0 to π, i.e., for the upper half; otherwise, obtain the values of the radius vector r for different values of θ in the range 0 to 2π.

Some standard curves

(a) $x^{2/3} + y^{2/3} = a^{2/3}$

This curve is known as 'Astroid'. The parametric equation of the curve may be written as $x = a\cos^3\theta, y = a\sin^3\theta$. The equation of given curve may be written as $\left(x^{1/3}\right)^2 + \left(y^{1/3}\right)^2 = a^{2/3}$. We see that replacing x by $-x$ and y by $-y$ does not change the equation of the curve, and therefore, the curve is symmetrical about both x and y axes. The curve meets the x-axis at the points $(a,0)$ and $(-a,0)$, and meets the y-axis at the points From the given equation, we can write $x = \left(a^{2/3} - y^{2/3}\right)^{3/2}$. It is clear that if $|y| > a$, then x becomes imaginary. Therefore, no part of the curve lies either above the line $y = a$ or below the line $y = -a$. Again, from the given equation, we can write $y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$. It is clear that if $|x| > a$, then y becomes imaginary. Therefore, no part of the curve lies either to the left of the line $x = -a$ or to the right of the line $x = a$. Thus, the complete curve can be drawn only if we can draw the portion of the curve lying in the first quadrant. For this, we compute the values of (x,y) corresponding to some values of θ in the range 0 to $\pi/2$ of the first quadrant, and present in the Table 1.

Table 1. Values of (x,y) corresponding to some values of θ

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
x	a	$\frac{3\sqrt{3}}{8}a$	$\frac{1}{2\sqrt{2}}a$	$\frac{1}{8}a$	0
y	0	$\frac{1}{8}a$	$\frac{1}{2\sqrt{2}}a$	$\frac{1}{2\sqrt{2}}a$	a

Thus, a rough sketch of the astroid is as shown in Fig. 1.

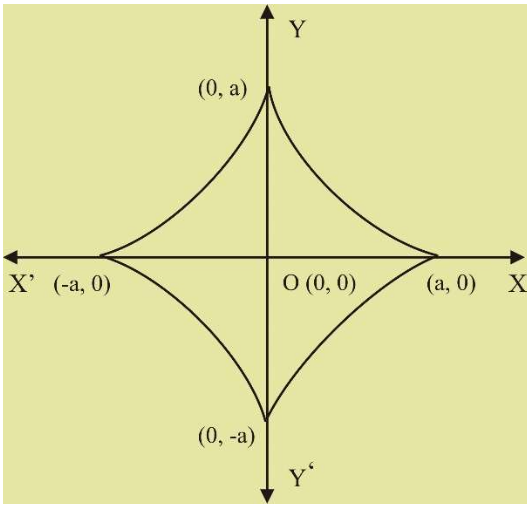

Fig. 1. A rough sketch of 'Astroid'

(b) $x^3 + y^3 = 3axy$

This curve is known as Folium of Descartes. This curve is symmetrical about the line $y = x$. The curve pass through the origin and meets the line $y = x$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$. The line $x + y + a = 0$ is the only asymptote of the curve (Ref. to Example of Chapter). A rough sketch of the curve is as shown in Fig. 2.

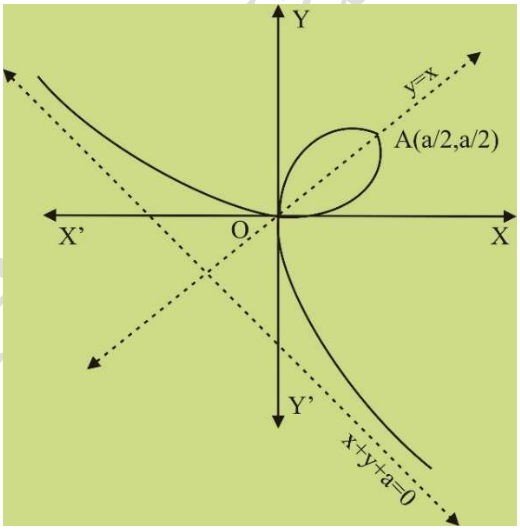

Fig. 2. A rough sketch of 'Folium of Descartes'

(c) $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$

This curve is known as Cycloid, and is traced by a fixed point taken on the rim of a cycle when the cycle is moved from one place to another on a plane surface without sliding. The sketch of the graph is as shown in Fig. 2. Here, θ varies from $-\pi$ to π.

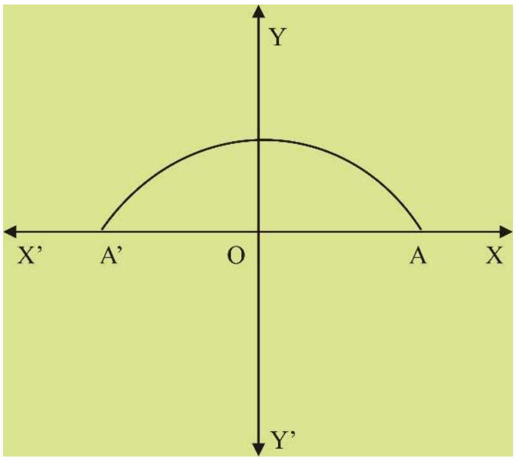

Fig. 3. A rough sketch of 'Folium of Descartes'

Note. Another form of cycloid:. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ Here, θ varies from 0 to 2π. The sketch of the graph is as shown in Fig. 3.

(d) $r = a(1 - \cos\theta)$

This curve is known as Cardioide. Since by replacing θ by $-\theta$, the equation of the curve remains unchanged, so the curve is symmetrical about the initial line. The values of radius vector r corresponding to some values of θ are given in Table 2. A rough sketch of the curve is as shown in Fig. 4.

Table 2. Length of radius vector r corresponding to some values of θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a\left(1 - \frac{\sqrt{3}}{2}\right)$	$a\left(1 - \frac{1}{\sqrt{2}}\right)$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

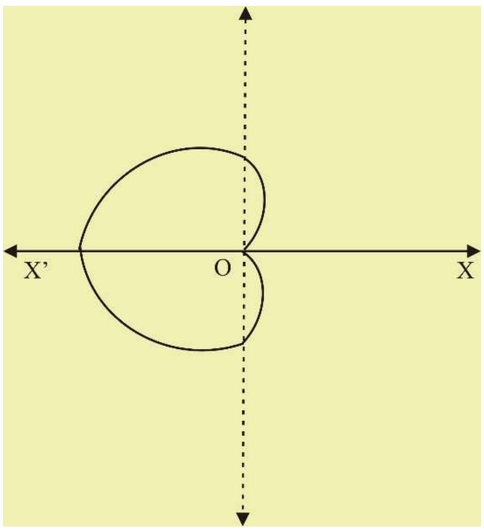

Fig. 4. A rough sketch of 'Cardioide'

Worked-out examples:

Example 1. Trace the curve represented by the equation $3ay^2 = x(x-a)^2$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the x-axis, but not symmetrical about the y-axis. The curve meets the x-axis at the points $(0,0)$ and $(a,0)$. The curve passes through the origin, and the tangent at origin is $x = 0$, i.e., y-axis. When $x < 0$, y^2 becomes negative, and hence, no part of the curve lies to the left of the line $x = 0$. But, as the value of x increases beyond a, the value of y also increases. Thus, a rough sketch of the curve is as shown in Fig. 5.

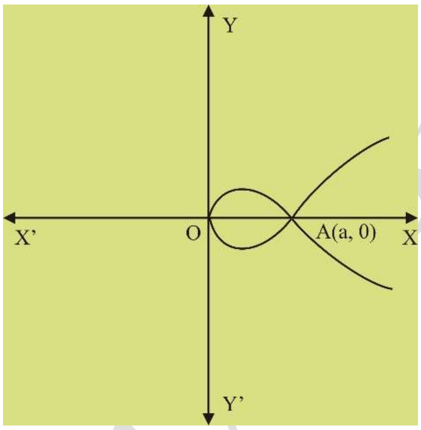

Fig. 5. A rough sketch of the curve given by the equation $3ay^2 = x(x-a)^2$

Example 2. Trace the curve represented by the equation $ay^2 = x^2(a-x)$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the x-axis. The curve meets the x-axis at the points $(0,0)$ and $(a,0)$. The curve passes through the origin, and the tangent at the origin is given by $ay^2 = ax^2$ (obtained by equating to zero, the lowest degree terms of the equation), i.e., $y = \pm x$. If we put some value of $x > a$, we see that y^2 becomes negative, which means that no part of the curve lies to the right of the line $x = a$. Thus, a rough sketch of the curve is as shown in Fig. 6.

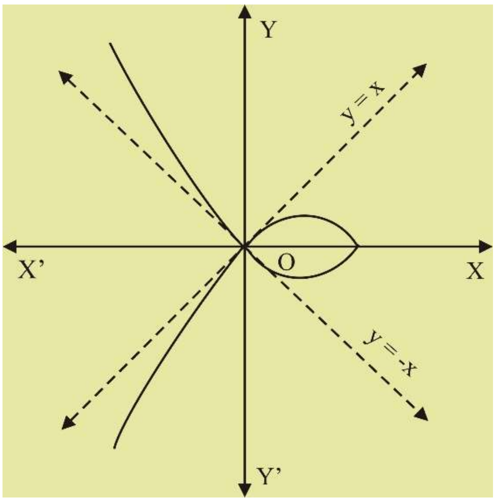

Fig. 6. A rough sketch of the curve given by the equation $ay^2 = x^2(a - x)$

Example 3. Trace the curve represented by the equation $y^2(2a - x) = x^3$, where $a > 0$ is a real constant.

Solution. The given curve is known as 'Cissoid of Diocles'. The curve is symmetrical about the x-axis. The curve meets the x-axis only at the origin. The curve passes through the origin, and the tangent at the origin is given by $y = 0$, i.e., x-axis. We see that when $x < 0$ or $x > 2a$, y^2 takes negative value, which means no part of the curve lies to the left of the line $x = 0$ or to the right of the line $x = 2a$. Moreover, the straight line $x = 2a$ is an asymptote of the given curve. Thus, a rough sketch of the curve is as shown in Fig. 7.

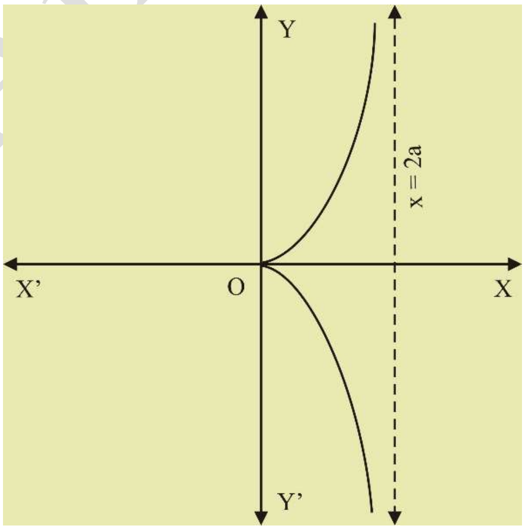

Fig. 7. A rough sketch of the curve given by the equation $ay^2 = x^2(a - x)$

Example 4. Trace the curve represented by the equation $x^{1/2} + y^{1/2} = a^{1/2}$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the line $y = x$. The curve meets the x-axis at the point $(a, 0)$ and meets the y-axis at $(0, a)$. From the given equation, we see that neither x nor y can take negative value, and hence, the curve lies in first quadrant only. Thus, a rough sketch of the curve is as shown in Fig. 8.

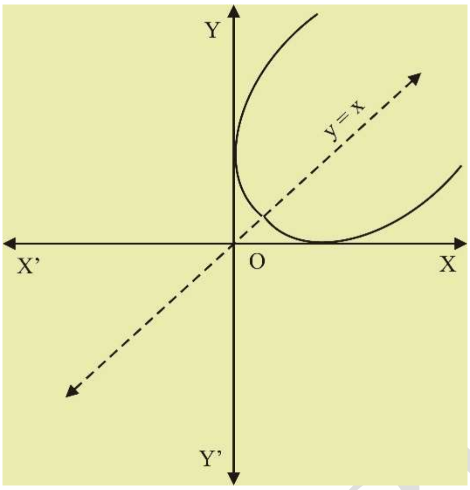

Fig. 8. A rough sketch of the curve given by the equation $x^{1/2} + y^{1/2} = a^{1/2}$

Example 5. Trace the curve represented by the equation $xy^2 + (x+a)^2(x+2a) = 0$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the x-axis, but not symmetrical about the y-axis. The curve meets the x-axis at the points $(-2a, 0)$ and $(-a, 0)$. The given equation may be re-written as $y^2 = -\frac{(x+a)^2(x+2a)}{x}$, which shows that if $x < -2a$ or $x > 0$, then y^2 becomes negative. Thus, no part of the curve lies to the left of the line Moreover, $x = 0$, i.e., y-axis is an asymptote to the curve. Considering all these points, a rough sketch of the curve is as shown in Fig. 9.

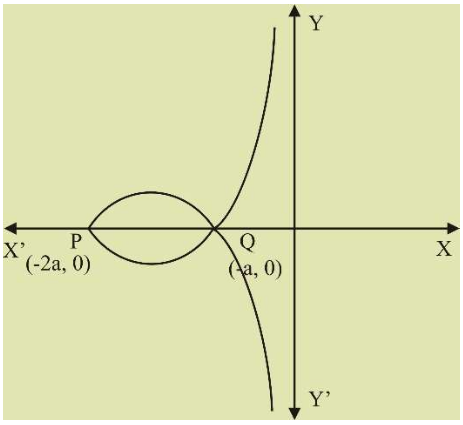

Fig. 9. A rough sketch of the curve given by the equation $xy^2 + (x+a)^2(x+2a) = 0$

Example 6. Trace the curve represented by the equation $a^2y^2 = x^2(a^2 - x^2)$, where $a > 0$ is a real constant.

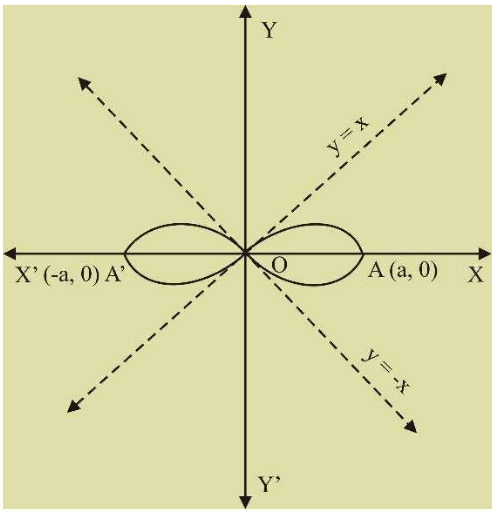

Fig. 10. A rough sketch of the curve given by the equation $a^2y^2 = x^2(a^2 - x^2)$

Solution. The curve is symmetrical about both the co-ordinate axes. The curve meets x-axis at the points $(-a, 0)$ and $(a, 0)$. The curve passes through the origin and $y = \pm x$ are the tangents at the origin. If $x > a$ or $x < -a$, then y^2 becomes negative, which means that no part of the curve lies to the left of the line $x = -a$ and to the right of the line $x = a$. Considering all these points, a rough sketch of the curve is as shown in Fig. 10.

Note. Bernoulli's Lemniscate Page no 23

Example 7. Trace the curve represented by the equation $y = \frac{a^3}{a^2+x^2}$, where $a > 0$ is a real constant.

Solution. The curve is symmetrical about the y-axis only. The curve meets y-axis at the point $(0, a)$. The equation of the curve may be re-written as $x^2 = \frac{a^2(a-y)}{y}$, which shows that if $y > a$, then x^2 takes negative value. Thus, no part of the curve lies above the line $y = a$. Moreover, $y = 0$ is an asymptote to the curve. Considering all these points, a rough sketch of the curve is as shown in Fig. 11.

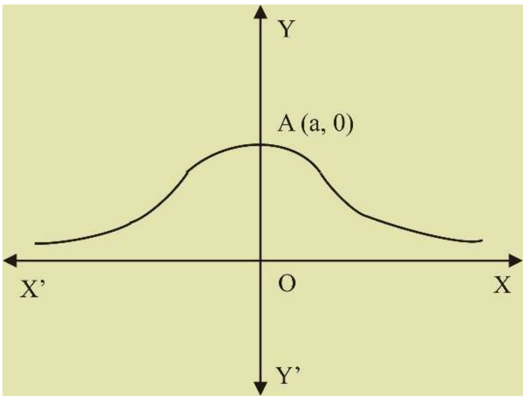

Fig. 11. A rough sketch of the curve given by the equation $y = \frac{a^3}{a^2+x^2}$

Example 8. Trace the curve represented by the equation $x^2y^2 = a^2(y^2 - x^2)$, where $a > 0$ is a real constant.

Solution. The given curve is symmetrical about both the co-ordinate axes. The curve meets x-axis only at the origin. The curve passes through the origin and $y = \pm x$ are the tangents at the origin. If $x > a$ or $x < -a$, then y^2 becomes negative, which means that no part of the curve lies to the left of the line $x = -a$ and to the right of the line $x = a$. Considering all these points, a rough sketch of the curve is as shown in Fig. 12.

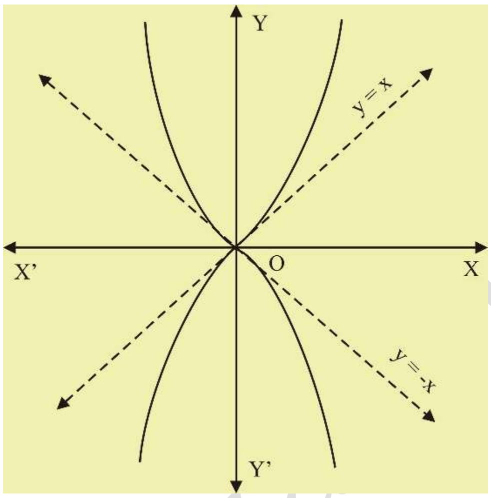

Fig. 12. A rough sketch of the curve given by the equation $x^2y^2 = a^2(y^2 - x^2)$

Example 9. Trace the curve represented by the equation $x^4 + y^4 = 2a^2xy$, where $a > 0$ is a real constant.

Solution. The given curve is symmetrical about the straight line $y = x$, and meets the line $y = x$ at the points (a, a) and $(-a, -a)$. The curve passes through the origin, and the tangents at the origin are $x = 0$ and $y = 0$. Considering all these points, a rough sketch of the curve is as shown in Fig. 13.

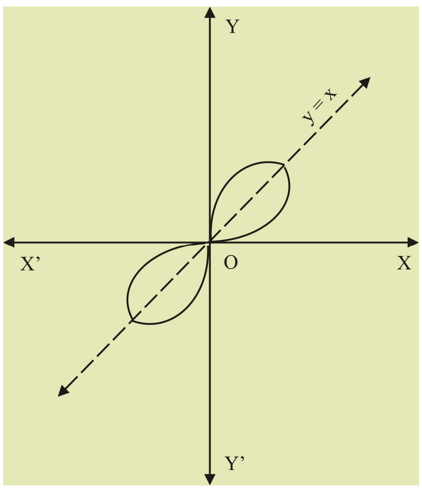

Fig. 12. A rough sketch of the curve given by the equation $x^4 + y^4 = 2a^2xy$

Exercises.

Trace the curves for the following equations.

1. $y^2(a-x) = x(x-b)^2$
2. $y^2(a+x) = x^2(b-x)$
3. $y^2 = (x-a)^3$
4. $9y^2 = (x+7)(x+4)^2$
5. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ (This curve is known as Astroid or Four Cusped Hypocycloid)

References.

1. Book: Application of Calculus: Theory and Problems by Sitansu Bandhopadhyay and Sunil Kumar Maity
2. Website: https://math.buet.ac.bd/public/faculty_profile/files/766768001.pdf